Eur. Phys. J. C 47, 513–524 (2006)
Digital Object Identifier (DOI) 10.1140/epjc/s2006-02571-1

THE EUROPEAN PHYSICAL JOURNAL C

Hopf solitons and Hopf Q-balls on S^3

C. Adam[1,a], J. Sánchez-Guillén[1,b], A. Wereszczyński[2,c]

[1] Departamento de Fisica de Particulas, Universidad de Santiago and Instituto Galego de Fisica de Altas Enerxias (IGFAE), 15782 Santiago de Compostela, Spain
[2] Institute of Physics, Jagiellonian University, Reymonta 4, 30-059 Kraków, Poland

Received: 21 February 2006 / Revised version: 8 April 2006 /
Published online: 7 June 2006 –

Abstract. Field theories with a S^2-valued unit vector field living on $S^3 \times \mathbb{R}$ space-time are investigated. The corresponding eikonal equation, which is known to provide an integrable sector for various sigma models in different spaces, is solved giving static as well as time-dependent multiply knotted configurations on S^3 with arbitrary values of the Hopf index. Using these results, we then find a set of hopfions with topological charge $Q_{\mathrm{H}} = m^2$, $m \in \mathbf{Z}$, in the integrable subsector of the pure CP^1 model. In addition, we show that the CP^1 model with a potential term provides time-dependent solitons. In the case of the so-called "new baby Skyrme" potential we find, e.g., exact stationary hopfions, i.e., topological Q-balls.
Our results further enable us to construct exact static and stationary Hopf solitons in the Faddeev–Niemi model with or without the new baby Skyrme potential. Generalizations for a large class of models are also discussed.

1 Introduction

Dynamical models allowing for stable knot-like structures seem to play an increasingly important role in modern physics. For instance, knotted solitons find some applications in condensed matter physics [1, 2] as topological defects in multi-component Bose condensates. On the other hand, in high energy physics the rising interest originates in the idea that glueballs, i.e., effective particle-like excitations in the low-energy limit of quantum gluodynamics, may be understood as closed, in general knotted, tubes of the squeezed color field (possibly due to the dual Meissner effect [3]). In fact, such a framework is in accordance with the standard picture of mesons where a quark and an antiquark are connected by a thin flux-tube of the gauge field. When the quark sources are absent, the ends of the tube must join to form a (in general knotted) loop. There has been made much effort to derive such a qualitative picture from the original quantum theory and to find the correct low-energy effective action with knotted solitons as stable excitations. One well-known proposal is the Faddeev–Niemi model [4], which is, in fact, just the S^2 restriction of the Skyrme model, as can be explicitly demonstrated [5]

$$\mathcal{L}_{\mathrm{FN}} = \frac{1}{2}\mu^2 \mathcal{L}_2 - \frac{1}{4e^2}\mathcal{L}_4 \,, \tag{1}$$

where

$$\mathcal{L}_2 \equiv (\partial_\mu \mathbf{n})^2 = 4\frac{\partial^\mu u \partial_\mu \bar{u}}{(1+u\bar{u})^2} \,, \tag{2}$$

$$\begin{aligned}\mathcal{L}_4 &\equiv [\mathbf{n}\cdot(\partial_\mu \mathbf{n} \times \partial_\nu n)]^2 \\ &= 8\frac{(\partial^\mu u \partial_\mu \bar{u})^2 - (\partial^\mu u \partial_\mu u)(\partial^\nu \bar{u} \partial_\nu \bar{u})}{(1+u\bar{u})^4} \,,\end{aligned} \tag{3}$$

where μ is a constant with the dimension of a mass, and e is a dimensionless constant. Further, n is a real three component unit vector field living in $(3+1)$ Minkowski space-time, and u is a complex scalar field related to the unit vector field by the standard stereographic projection

$$n = \frac{1}{1+|u|^2}(u+\bar{u}, -\mathrm{i}(u-\bar{u}), |u|^2 - 1). \tag{4}$$

There are some arguments that this field, connected with the primary gauge field via the Cho–Faddeev–Niemi decompositions, might describe the infrared relevant degrees of freedom of quantum gluodynamics [6–10]. On the other hand, the stability of the spectrum and even of the field decomposition under quantum fluctuations is a matter of active research and discussion [11, 12].

It has been proved that the Faddeev–Niemi model indeed supports knotted solitons [13] with a non-zero value of the Hopf index $Q_H \in \pi_3(S^2)$. However, only numerical solutions have been reported [14, 15] and many important questions concerning, e.g., the geometry of the stable (or meta-stable) configurations in a fixed topological sector are still unsolved. In order to understand the behavior of hopfions in an analytical way and to test some ideas borrowed

[a] e-mail: adam@fpaxp1.usc.es
[b] e-mail: joaquin@fpaxp1.usc.es
[c] e-mail: wereszczynski@th.if.uj.edu.pl

from other soliton systems, two dynamical models have been proposed. They are known as the Nicole model [16], described by

$$\mathcal{L}_{\rm Ni} = \frac{1}{2}\mathcal{L}_2^{\frac{3}{2}}\,, \tag{5}$$

and the Aratyn–Ferreira–Zimerman model [17], with Lagrangian

$$\mathcal{L}_{\rm AFZ} = \frac{1}{4}\mathcal{L}_4^{\frac{3}{4}}\,. \tag{6}$$

A common feature of these two non-linear models is their invariance under scale reparametrizations. This provides a new way (originally proposed by Deser et al. [18]) to circumvent Derrick's theorem. In addition, they possess exact soliton solutions.

However, there is a different strategy to construct exact hopfions. Namely, it is possible to change the base space in such a manner that the topological content of the theory remains unchanged. The most obvious proposition is to investigate fields on a three-dimensional sphere $S^3_{R_0}$, where R_0 is its radius, instead of the standard three-dimensional Euclidean space $\mathbb{R}^3$ [19, 20]. In this case, the introduction of the new parameter R_0 sets the scale in the model and therefore gives an alternative way to circumvent Derrick's theorem about the non-existence of static solitons. Hopf solitons on S^3 have been recently considered by Ward and Ferreira et al. in the context of Faddeev–Niemi- [19] and AFZ-like models [20], respectively. In the present paper, we would like to further develop these investigations. Concretely, in Sect. 2 we construct a family of solutions for the static as well as for the time-dependent eikonal equation for arbitrary values of the Hopf index. The eikonal equation defines integrable subsectors for the models discussed in the subsequent sections, and its solutions, therefore, will help us in finding explicit soliton solutions. In Sect. 3 we study the $CP(1)$ model and find a class of static solutions and, when a potential is added, a similar class of time-dependent solutions. The same results can be found for a family of generalized $CP(1)$ models. In Sect. 4 we study another family of models, among which the Faddeev–Niemi model can be found. We establish the existence of a Hopf soliton with topological charge one for all of them. Further, for the Faddeev–Niemi model with and without a potential term, the existence of a time-dependent, stationary solution is demonstrated. In addition, we construct some generalizations of these models, which have solitons with higher Hopf index. Section 5 contains our conclusions.

To finish the introduction, let us remind the reader of some details of the geometry of the three-sphere. A three-sphere S^3 with radius R_0 embedded in four-dimensional Euclidean space $\mathbb{R}^4$ is described by the equation

$$X_1^2+X_2^2+X_3^2+X_4^2 = R_0^2\,, \tag{7}$$

where the X_i are the usual orthonormal coordinates in $\mathbb{R}^4$. Further, the metric on the surface defined by (7) is induced by the standard Euclidean metric on $\mathbb{R}^4$. Introducing coordinates on S^3 as in [20],

$$\begin{aligned} X_1 &= R_0\sqrt{z}\cos\phi_2\,, \quad X_3 = R_0\sqrt{1-z}\cos\phi_1\,,\\ X_2 &= R_0\sqrt{z}\sin\phi_2\,, \quad X_4 = R_0\sqrt{1-z}\sin\phi_1\,, \end{aligned} \tag{8}$$

where $z\in[0,1]$ and the angles $\phi_1,\phi_2\in[0,2\pi]$, the metric and volume form on space-time $\mathbb{R}\times S^3$ are

$$\mathrm{d}s^2 = \mathrm{d}t^2 - R_0^2\left(\frac{\mathrm{d}z^2}{4z(1-z)} + (1-z)\,\mathrm{d}\phi_1^2 + z\,\mathrm{d}\phi_2^2\right)\,, \tag{9}$$

$$\mathrm{d}V = \frac{1}{2}\,\mathrm{d}t\,\mathrm{d}z\,\mathrm{d}\phi_1\,\mathrm{d}\phi_2\,. \tag{10}$$

2 Eikonal knots on S^3

The construction of soliton solutions with non-zero Hopf index is sometimes facilitated by restricting the original theory to an integrable submodel [21], where integrability is understood as the existence of an infinite number of local conserved currents. In typical situations (Nicole or Faddeev–Niemi model), such an integrable subsystem can be defined by imposing the complex eikonal equation [22][1]

$$(\partial_\mu u)^2 = 0\,. \tag{11}$$

Solutions of the complex eikonal equation on $\mathbb{R}^3$ describe (linked) torus knots with an arbitrary value of the Hopf charge [25, 26] and in some particular cases may help to derive hopfions in dynamical systems [27]. On the other hand, the fact that the eikonal equation is an integrability condition for sigma models does not depend on the base space. Thus, as our aim is to study knotted configurations on S^3, it is important to solve the eikonal on the three-sphere as well. In fact, all solitons which we study will obey the eikonal equation as well.

Let us assume the following static Ansatz [17, 20]:

$$u_0 = f(z)\mathrm{e}^{\mathrm{i}(m_1\phi_1+m_2\phi_2)}\,, \tag{12}$$

where m_1, m_2 are integer numbers. Then we find

$$\begin{aligned}\nabla u =& \frac{1}{R_0}\left[2\sqrt{z(1-z)}f'\hat{e}_z + \frac{im_1 f}{\sqrt{1-z}}\hat{e}_{\phi_1} + \frac{im_2 f}{\sqrt{z}}\hat{e}_{\phi_2}\right]\\ &\times \mathrm{e}^{\mathrm{i}(m_1\phi_1+m_2\phi_2)}\,, \end{aligned} \tag{13}$$

and (11) can be rewritten as follows:

$$4z(1-z)f'^2 - \frac{f^2}{z(1-z)}\left(zm_1^2+(1-z)m_2^2\right) = 0\,, \tag{14}$$

or

$$\frac{f'^2}{f^2} = \frac{zm_1^2+(1-z)m_2^2}{4z^2(1-z)^2}\,. \tag{15}$$

[1] For some Lagrangian-dependent "generalizations" of the eikonal equation and their application to integrability, see [23]. Moreover, a weaker integrability condition has been investigated in [24].

One can solve this equation and obtain the following solutions:

$$f_{\pm} = C \left[\left(\frac{m_1 + \sqrt{(m_1^2 - m_2^2)z + m_2^2}}{m_1 - \sqrt{(m_1^2 - m_2^2)z + m_2^2}} \right)^{m_1} \times \left(\frac{-m_2 + \sqrt{(m_1^2 - m_2^2)z + m_2^2}}{m_2 + \sqrt{(m_1^2 - m_2^2)z + m_2^2}} \right)^{m_2} \right]^{\pm\frac{1}{2}} . \tag{16}$$

Here C is a complex constant. Our solutions simplify a lot if we assume the special case $m_1 = m_2 = m$. Then

$$f_{\pm} = C \left(\frac{1}{z} - 1 \right)^{\pm\frac{m}{2}} . \tag{17}$$

The point is that such profile functions, if inserted into the Ansatz, give configurations with a non-trivial value of the pertinent topological charge. Namely [20],

$$Q_{\mathrm{H}} = \pm m_1 m_2 . \tag{18}$$

Moreover, taking into account the symmetries of the complex eikonal equation we can find more general solutions

$$u = F(u_0) , \tag{19}$$

where F is any (anti-) holomorphic function of the basic solution u_0. Thus, we can conclude that the complex eikonal equation on S^3 describes linked torus knots, as its counterpart on R^3.

The eikonal equation on S^3 also enables us to obtain time-dependent knotted configurations, unlike the standard $\mathbb{R}^3$ case where no time-dependent solutions are known. The non-static eikonal equation has the form

$$(\partial_t u)^2 - (\nabla u)^2 = 0 . \tag{20}$$

Let us assume that the time-dependence can be factorized. Due to (20) such a factorization must have the form of an exponential,

$$u_0 = f(z) \mathrm{e}^{\pm\lambda t} \mathrm{e}^{\mathrm{i}(m_1\phi_1 + m_2\phi_2)} , \tag{21}$$

where λ is a complex parameter and $f(z)$ is a new profile function yet to be determined. It is straightforward to notice that there are two generic situations.

First of all, for $\lambda \in R$ we can find exploding or collapsing solutions, depending on the sign of the parameter. Now, formula 20 takes the form

$$4z(1-z)f'^2 - f^2 \left[\frac{zm_1^2 + (1-z)m_2^2}{z(1-z)} + R_0^2 \lambda^2 \right] = 0 \tag{22}$$

or

$$\frac{f'^2}{f^2} = \frac{1}{4} \left[\frac{zm_1^2 + (1-z)m_2^2}{z^2(1-z)^2} + \frac{R_0^2 \lambda^2}{z(1-z)} \right] . \tag{23}$$

This equation can be integrated giving exact but rather complicated solutions for the shape function

$$\begin{aligned} f_{\pm}(z) = C \\ \times \exp\left[\mp \frac{1}{2a} \arctan\left(\frac{1 + a^2 \left(m_1^2 - m_2^2\right) - 2z}{2\sqrt{z(1-z) + a^2(m_2^2(1-z) + m_1^2 z)}} \right) \right] \\ \times \left[\frac{1}{a^3 m_2^3 z} \left(-z + a^2 \left(m_2^2(-2+z) - m_1^2 z \right) + 2am_2 \sqrt{z(1-z) + a^2\left(m_2^2(1-z) + m_1^2 z\right)} \right) \right]^{\pm\frac{m_2}{2}} \\ \times \left[\frac{1}{a^3 m_1^3 (-1+z)} \left(1 - z + a^2 \left(m_2^2(1-z) - m_1^2(1+z) \right) - 2am_1 \sqrt{z(1-z) + a^2\left(m_2^2(1-z) + m_1^2 z\right)} \right) \right]^{\mp\frac{m_1}{2}} , \end{aligned} \tag{24}$$

where C is a complex constant and

$$a^2 = \frac{1}{R_0^2 \lambda^2} . \tag{25}$$

It is easy to see that these new profile functions are asymptotically (for $z \to 0$ and $z \to 1$) identical to their static counterparts. The additional term in the eikonal equation only modifies the behavior in the intermediate region. Therefore, the topological features of the time-dependent solutions are analogous to the static case.

Our solutions take a simpler form if $m_1 = m_2 = m$,

$$f_{\pm}(z) = C \exp\left[\mp \frac{1}{2a} \arctan\left(\frac{1-2z}{2\sqrt{m^2 a^2 + z - z^2}} \right) \right] \times \left(\frac{1-z}{z} \cdot \frac{z + 2ma\left(ma + \sqrt{m^2 a^2 + z - z^2}\right)}{1 - z + 2ma\left(ma + \sqrt{m^2 a^2 + z - z^2}\right)} \right)^{\mp\frac{m}{2}} . \tag{26}$$

Another type of time-dependent solutions is a family of time-periodic configurations. Now $\lambda = \mathrm{i}\omega$, where $\omega \in R$. Thus,

$$u_0 = f(z) \mathrm{e}^{\pm\mathrm{i}\omega t} \mathrm{e}^{\mathrm{i}(m_1\phi_1 + m_2\phi_2)} , \tag{27}$$

where the unknown shape function satisfies the following equation:

$$\frac{f'^2}{f^2} = \frac{1}{4} \left[\frac{zm_1^2 + (1-z)m_2^2}{z^2(1-z)^2} - \frac{R_0^2 \omega^2}{z(1-z)} \right] . \tag{28}$$

In this case the solution is

$$f_\pm(z) = C\Big(-1+a^2(m_1^2-m_2^2)+2z + 2\sqrt{a^2\big(m_2^2(1-z)+m_1^2 z\big)-z(1-z)}\Big)^{\mp\frac{1}{2a}} \times\left[\frac{1}{a^3m_1^3(-1+z)} \times\Big(-1+z+a^2\big(m_2^2(1-z)+m_1^2(1+z)\big) + 2\sqrt{a^2\big(m_2^2(1-z)+m_1^2 z\big)-z(1-z)}\Big)\right]^{\pm\frac{m_1}{2}} \times\left[\frac{1}{a^3m_2^3 z}\Big(z+a^2(m_2^2(-2+z)-m_1^2 z) - 2\sqrt{a^2\big(m_2^2(1-z)+m_1^2 z\big)-z(1-z)}\Big)\right]^{\mp\frac{m_2}{2}}, \tag{29}$$

or in the simpler case, when $m_1=m_2=m$,

$$f_\pm(z) = C\left(-1+2z+2\sqrt{m^2a^2-z(1-z)}\right)^{\pm\frac{1}{2a}} \times\left(\frac{-1+z}{z}\cdot\frac{z-2ma(ma+\sqrt{m^2a^2-z(1-z)})}{-1+z+2ma(ma+\sqrt{ma^2-z(1-z)})}\right)^{\pm\frac{m}{2}}. \tag{30}$$

Such shape functions give non-trivial topological configurations if f is a smooth, real function which tends to 0 for $z\to 0$ and to ∞ when $z\to 1$ (or inversely). Therefore we get a restriction for the frequencies of the stationary solutions with a fixed topological charge:

$$\omega^2 \le \frac{2\left(m_1^2+m_2^2\right)}{R_0^2}. \tag{31}$$

In other words, there is an upper bound for the frequencies of a stationary solution. Only configurations with lower frequencies can be constructed. Equation (31) leads to two important observations. Firstly, the range of possible frequencies grows with the topological charge. Knots with higher topological charge can have higher frequencies. Secondly, the range becomes narrower if the radius of the base space grows. Thus, for the Euclidean space, i.e., when $R\to\infty$, no time-periodic eikonal knots can be found. At the end of this section let us notice that the complex eikonal equation admits also topologically trivial solutions. An interesting example can be found if we assume that the complex field is a function only of time and z variable. Then using the method of characteristics we derive the following general solution:

$$u = u\left(t\pm\frac{1}{2}\arcsin(1-2z)\right). \tag{32}$$

One can immediately see that such a solution describes a very non-linear travelling wave.

3 CP^1 model on S^3

So far, the considered knots have been only solutions of the complex eikonal equation, without any underlying Lagrange structure. In the next sections we show that at least some of the eikonal knots appear as solutions of a large family of non-linear sigma models on S^3. Let us mention at this point that for the purely quartic model with Lagrangian $\mathcal{L}_4$ – which is integrable without any additional constraint – both static and time-dependent, stationary hopfions on S^3 have been found and studied in [20].

3.1 Static solutions

Let us start with the simplest example, i.e., the CP^1 model,

$$\mathcal{L}_{CP^1} \equiv \frac{1}{4}\mathcal{L}_2 = \frac{\partial_\mu u\partial^\mu\bar{u}}{(1+|u|^2)^2}. \tag{33}$$

The equation of motion reads

$$\partial_\mu^2 u - \frac{2\bar{u}}{1+|u|^2}(\partial_\mu u)^2 = 0. \tag{34}$$

This equation is certainly satisfied for a submodel, where the complex field u obeys the two equations

$$\partial_\mu^2 u = 0 \quad\text{and}\quad (\partial_\mu u)^2 = 0, \tag{35}$$

i.e., the wave equation and the eikonal equation.[2] Due to the fact that the eikonal equation is imposed, this submodel belongs to the integrable systems.

In order to find static knotted configurations we assume the same Ansatz as in (12). Then

$$\nabla^2 u = \frac{1}{R_0^2}\left[4\partial_z\left(z(1-z)f'\right) - f\left(\frac{m_1^2}{1-z}+\frac{m_2^2}{z}\right)\right] \times \mathrm{e}^{\mathrm{i}(m_1\phi_1+m_2\phi_2)}, \tag{36}$$

and the first equation in (35) can be rewritten as

$$4\partial_z\left(z(1-z)f'\right) = f\left(\frac{m_1^2}{1-z}+\frac{m_2^2}{z}\right). \tag{37}$$

Of course, as the subsystem consists of the static eikonal equation, as well, the profile function f has to satisfy (15). Therefore, the left hand side of (37) can be expressed by (15). Then we obtain

$$4\partial_z\left(z(1-z)f'\right) = 4z(1-z)f'\frac{f'}{f}. \tag{38}$$

A first integration leads to

$$\ln\left(\frac{1}{b}\frac{z(1-z)f'}{f}\right) = 0, \tag{39}$$

[2] This pair of equations has been studied first, in the context of the CP^1 model in 2+1 dimensional space-time, in [28].

which possesses the following solutions:

$$f = B\left(\frac{1}{z} - 1\right)^b . \qquad (40)$$

Here B and b are arbitrary, in general complex constants. However, as this solution should satisfy also the eikonal equation, we find that $b = \pm m/2$. Therefore, the field configurations

$$u_\pm = B\left(\frac{1}{z} - 1\right)^{\pm\frac{m}{2}} \mathrm{e}^{\mathrm{i}m(\phi_1+\phi_2)} \qquad (41)$$

are solutions of the submodel (35) and, as a consequence, they are static solutions of CP^1 model. Moreover, they carry a non-zero value of the Hopf index

$$Q_\mathrm{H} = \pm m^2 . \qquad (42)$$

More complicated solutions can be constructed if we take advantage of the symmetries of the submodel. In fact, it is easily checked that any $\tilde{u}$ of the form

$$\tilde{u} = F(u) \qquad (43)$$

is a solution, where F is any (anti-) holomorphic function, and u ($\bar{u}$) is a solution of the submodel (e.g., of the form (41) derived above). Thus, we can obtain quite complicated linked configurations with arbitrary Hopf charge. Let us now calculate the energies of the obtained solutions. They are given by

$$E = \int_{S^3} \frac{\nabla u \nabla \bar{u}}{(1+|u|^2)^2} \frac{1}{2} \mathrm{d}z\, \mathrm{d}\phi_1 d\phi_2 . \qquad (44)$$

Thus, inserting (41) we get

$$E = 4\pi^2 m^2 R_0 \int_0^1 \frac{1}{z(1-z)} \frac{f^2}{(1+f^2)^2} \mathrm{d}z , \qquad (45)$$

and finally

$$E = 4\pi^2 R_0 |m| . \qquad (46)$$

Re-introducing the Hopf index we therefore find

$$E = 4\pi^2 R_0 |Q_\mathrm{H}|^{\frac{1}{2}} , \qquad (47)$$

i.e., the energies grow like the square root of the Hopf index. One can also observe that there is some degeneracy in the energy spectrum, because the energy remains the same for all values of the parameter B. As we will discuss in the last section, these solutions are not stable but rather saddle point solutions.

3.2 Time-dependent solutions

Exact, time-depended hopfions can be found if we consider the CP^1 model with a potential explicitly breaking the global $O(3)$ symmetry:

$$\mathcal{L} = \frac{1}{4}\mathcal{L}_2 - V_\mathrm{I} \equiv \frac{\partial_\mu u \partial^\mu \bar{u}}{(1+|u|^2)^2} - \frac{\beta^2}{4}\frac{|u|^2}{(1+|u|^2)^2} . \qquad (48)$$

Here β^2 is a positive constant. Such a potential has a very simple form if we express it in terms of the original unit vector field ,

$$V_\mathrm{I}(\mathbf{n}) = \frac{\beta^2}{16}\left[1-(n^3)^2\right] . \qquad (49)$$

It is worth mentioning that this potential has been previously considered in the context of the Skyrme model on the plane. More precisely, it is the potential part of the so-called “new baby Skyrme model” [29], which stabilizes the topological solitons in that model. In addition, possible applications of sigma-model type theories to the low-energy sector of YM theory require the explicit breaking of the global $O(3)$ symmetry, which may be achieved, e.g., by the introduction of a symmetry-breaking potential like the one chosen above [31, 32] (see also our remark in the summary section).

The equation of motion for our model is

$$\frac{1}{(1+|u|^2)^2}\partial_\mu\partial^\mu u - \frac{2\bar{u}}{(1+|u|^2)^3}(\partial u)^2 + \frac{\beta^2}{4}\frac{u}{(1+|u|^2)^2} - \frac{\beta^2 u|u|^2}{2(1+|u|^2)^3} = 0 . \qquad (50)$$

Similarly as in the pure CP^1 model it is possible to define a submodel consisting of two relative simple equations: a dynamical one,

$$\partial_\mu\partial^\mu u + \frac{\beta^2}{4} u = 0 , \qquad (51)$$

and a constraint being a modification of the eikonal equation

$$2\bar{u}(\partial_\mu u)^2 + \frac{\beta^2}{2} u|u|^2 = 0 . \qquad (52)$$

Obviously, every solution of the subsystem obeys the equation of motion for the full model. Notice that such a submodel is a “massive” modification of the pure CP^1 submodel with a “imaginary mass”. In particular, formula (52) can be rewritten in the form of the massive eikonal equation [30]

$$(\partial_\mu u)^2 - M^2 u^2 = 0 , \qquad (53)$$

where the “mass” parameter is $M^2 = -\beta^2/4$.

Once again solutions of the submodel (51) and (52) are assumed to have the form

$$u = f(z)\mathrm{e}^{\pm\mathrm{i}\omega t}\mathrm{e}^{\mathrm{i}(m_1\phi_1 + m_2\phi_2)} . \qquad (54)$$

Then, we derive the following equations for the unknown shape function:

$$4\partial_z\left(z(1-z)f'\right) - f\left[\frac{m_1}{1-z} + \frac{m_2}{z} - R_0^2\omega^2 + \frac{\beta^2 R_0^2}{4}\right] = 0 , \qquad (55)$$

$$4z(1-z)f'^2 - f^2\left[\frac{m_1}{1-z} + \frac{m_2}{z} - R_0^2\omega^2 + \frac{\beta^2 R_0^2}{4}\right] = 0 . \qquad (56)$$

These expressions can be simplified if we impose an additional condition for the frequency of the stationary solutions

$$\omega^2 = \frac{\beta^2}{4}\,. \tag{57}$$

Then we get a set of equations which are identical to the static equations in the pure CP^1 model. Therefore the shape function is given by

$$f(z) = C\left(\frac{1}{z} - 1\right)^{\pm\frac{m}{2}}, \tag{58}$$

where $m = m_1 = m_2$ and C is a complex constant. To summarize, we have found a family of topologically non-trivial, stationary hopfions

$$u = C\left(\frac{1}{z} - 1\right)^{\pm\frac{m}{2}} \mathrm{e}^{\pm \mathrm{i}\frac{\beta}{2}t} \mathrm{e}^{\mathrm{i}m(\phi_1+\phi_2)}\,. \tag{59}$$

Such stationary configurations which, although they rotate in an internal space, possess a time-independent energy density, are known as Q-balls. They provide, e.g., a well-known example of non-topological solitons. In the non-topological case these objects normally carry a conserved charge where the conserved current is a Noether current originating from an unbroken continuous global symmetry. In our case it is the remaining unbroken $O(2)$ subgroup of $O(3)$. Our solutions, however, have a conserved topological charge in addition to the non-topological Noether charge

$$Q = \mathrm{i}\int \frac{\bar{u}\partial_t u - u\partial_t \bar{u}}{(1+|u|^2)^2}\,\mathrm{d}V\,. \tag{60}$$

The energy of the stationary solutions reads

$$E = \int \mathrm{d}V\left(\frac{\nabla u \nabla \bar{u}}{(1+|u|^2)^2} + \frac{\partial_t u \partial_t \bar{u}}{(1+|u|^2)^2} + \frac{\beta^2}{4}\frac{|u|^2}{(1+|u|^2)^2}\right). \tag{61}$$

Thus

$$E = 4\pi^2 R_0 |Q_{\mathrm{H}}|^{\frac{1}{2}} + \frac{1}{2}|\beta Q|\,, \tag{62}$$

where (46) has been taken into account. As one might have expected, the Q-hopfions modify the standard CP^1 modeline in such a way that the degeneracy in the energy is lifted.

It should be noticed that analogous stationary solutions of the CP^1 model with the new baby Skyrme potential living in $(2+1)$ dimensional Minkowski space-time and carrying the pertinent topological charge (winding number) have been previously found by Leese [33]. They are known as Q-lumps.

Let us mention an interesting difference between Q-lumps and Q-hopfions. Q-hopfions have finite Noether charge and finite energy for all values of the topological Hopf charge, including Hopf charge one. On the other hand, it has been shown that in $(2+1)$ dimensions the energy of a Q-lump configuration is finite if and only if the topological charge is at least two [33].

Another type of time-dependent configurations can be obtained in the CP^1 model with a different kind of potential:

$$\mathcal{L} = \frac{1}{4}\mathcal{L}_2 - V_{\mathrm{II}} \equiv \frac{\partial_\mu u \partial^\mu \bar{u}}{(1+|u|^2)^2} - \frac{\beta^2}{16}\left(\frac{1-|u|^2}{1+|u|^2}\right)^2\,. \tag{63}$$

This potential also takes an elegant form if expressed by the unit vector field, namely

$$V_{\mathrm{II}}(\mathbf{n}) = \frac{\beta^2}{16}(n^3)^2\,. \tag{64}$$

In this case we obtain another massive modification of the free CP^1 submodel with a real mass $M^2 = \beta^2/4$.

One can check that now the time-dependent solutions are given by

$$u = C\left(\frac{1}{z} - 1\right)^{\pm\frac{m}{2}} \mathrm{e}^{\pm\frac{\beta}{2}t} \mathrm{e}^{\mathrm{i}m(\phi_1+\phi_2)}\,, \tag{65}$$

describing collapsing or exploding unknots.

In spite of the fact that our time-dependent hopfions are not sensitive to the radius of the sphere R_0, their energy is. Thus, such solutions do not lead to finite energy configurations in the limit $R_0 \to \infty$, that is, in three-dimensional Euclidean space.

Finally, let us notice that, contrary to the pure CP^1 model, a superposition of dynamical solutions derived for the submodels (51) and (52) is no longer a solution. This is due to the non-linearity of the constraint (52). However, we can obtain time-dependent multi-knotted solutions if we assume that such a configuration moves collectively. That is to say, the general solution is

$$u = F(u_{\mathrm{s}})\mathrm{e}^{\pm \mathrm{i}\omega t}\,, \tag{66}$$

where u_{s} is an arbitrary static solution of the pure CP^1 model and F is any (anti-) holomorphic function.

3.3 Generalized CP^1 models

The obtained results may be easily generalized to more complicated models. Namely, let us consider the following family of Lagrangians:

$$\mathcal{L} = \sigma(|u|^2)\,\frac{\partial_\mu u \partial^\mu \bar{u}}{(1+|u|^2)^2}\,, \tag{67}$$

where $\sigma(|u|^2)$ is any function of the modulus squared. This family represents CP^1 models with a "dielectric" function σ. The equation of motion reads

$$\tilde{\sigma}\partial_\mu\partial^\mu u + \tilde{\sigma}'\bar{u}\partial_\mu u \partial^\mu u = 0\,, \tag{68}$$

where $\tilde{\sigma} \equiv \sigma/(1+|u|^2)^2$, and the prime denotes the derivative with respect to the argument $|u|^2$. Thus, we get that solutions of the submodel (35) also obey (68) and, as a consequence, all generalized models possess the same static solutions given by (41).

Similarly, time-dependent solutions can be derived if we consider the generalized CP^1 models with a potential:

$$\mathcal{L} = \sigma(|u|^2)\left(\frac{\partial_\mu u \partial^\mu \bar{u}}{(1+|u|^2)^2} - \frac{\beta^2}{4}\frac{|u|^2}{(1+|u|^2)^2}\right). \qquad (69)$$

The field equation is

$$\tilde{\sigma}'\bar{u}\left((\partial_\mu u)^2 - \frac{\beta^2}{4}\right) + \tilde{\sigma}\left(\partial_\mu^2 u - \frac{\beta^2}{4}u\right) = 0. \qquad (70)$$

Therefore, solutions of the dynamical subsystem of the CP^1 model (51) and (52) satisfy (70) as well. That is to say, we have shown that the system (69) possesses stationary hopfions,

$$u = C\left(\frac{1}{z} - 1\right)^{\pm\frac{m}{2}} \mathrm{e}^{\pm \mathrm{i}m(\phi_1+\phi_2)}\mathrm{e}^{\pm \mathrm{i}\omega t}, \qquad (71)$$

where the frequency obeys the relation $\omega^2 = \beta^2/4$, as before. As an interesting example, let us mention a model with the so-called "old baby Skyrme" potential,

$$\mathcal{L} = \frac{(\partial \mathbf{n})^2}{1+n^3} - \frac{\beta^2}{16}(1-n^3). \qquad (72)$$

Analogously, one can construct Lagrangians which possess exact collapsing/exploding time solutions (65),

$$\mathcal{L} = \sigma(|u|^2)\left(\frac{\partial_\mu u \partial^\mu \bar{u}}{(1+|u|^2)^2} - \frac{\beta^2}{4}\left(\frac{1-|u|^2}{1+|u|^2}\right)^2\right). \qquad (73)$$

It is straightforward to obtain time-dependent multiknotted configurations.

4 Other models

4.1 Hopfion with $Q_\mathrm{H} = 1$

The aim of this section is to investigate a rather general family of non-linear sigma models on S^3. The unique restriction which we assume is that the Lagrange density is any reasonable function of the quantity

$$l = \frac{1}{4}\mathcal{L}_2 = \frac{\partial_\mu u \partial^\mu \bar{u}}{(1+|u|^2)^2}. \qquad (74)$$

Thus, we will analyze the following models:

$$\mathcal{L} = \mathcal{L}\left(\frac{\partial_\mu u \partial^\mu \bar{u}}{(1+|u|^2)^2}\right). \qquad (75)$$

One well-known member of that family is the Nicole model $\mathcal{L}_{\mathrm{Ni}} = 4l^{3/2}$. Of course, the pure CP^1 model belongs to this family as well. However, as this case is rather special, we have discussed it separately in the previous section.

The equation of motion reads

$$\partial_\mu\left(\frac{\mathcal{L}'}{(1+|u|^2)^2}\partial^\mu u\right) + \frac{2u}{(1+|u|^2)^3}\mathcal{L}'(\partial_\mu u \partial^\mu \bar{u}) = 0, \quad (76)$$

or

$$\mathcal{L}'\partial_\mu\partial^\mu u + \mathcal{L}''(\partial_\mu l\, \partial^\mu u) - \frac{2\bar{u}}{1+|u|^2}\mathcal{L}'(\partial_\mu u)^2 = 0, \qquad (77)$$

where the prime denotes the differentiation with respect to l. Analogously as for the pure CP^1 model it is possible to define an integrable submodel:

$$\partial_\mu^2 u = 0 \quad (\partial_\mu u)^2 = 0 \quad \text{and} \quad \partial_\mu l\, \partial^\mu u = 0. \qquad (78)$$

As we see, such a subsystem is a restriction of the submodel for the pure CP^1 model (35), where the scalar field must obey an additional equation. Therefore, a static soliton solution in this submodel can be derived if we impose the additional condition $\partial_\mu l\, \partial^\mu u = 0$ on the solutions of the pure CP^1 model obtained above.

We calculate

$$l = \frac{1}{(1+f^2)^2}\frac{1}{R_0^2}\left(4z(1-z)f'^2 + \frac{f^2\left(m_1^2 z + (1-z)m_2^2\right)}{z(1-z)}\right) \qquad (79)$$

or, if we put $m_1 = m_2 = m$,

$$l = \frac{2f^2m^2}{z(1-z)R_0^2}\frac{1}{(1+f^2)^2} = \frac{2m^2}{z(1-z)R_0^2}\frac{\left(\frac{1}{z}-1\right)^m}{\left[1+\left(\frac{1}{z}-1\right)^m\right]^2}. \qquad (80)$$

We immediately notice that for $m = 1$ we get $l = 2/R_0^2 =$ const. and the additional condition is trivially obeyed. That means that we have constructed a solution of the submodel (78), i.e. a topological solution of the family of models (75) with unit Hopf index $Q_\mathrm{H} = 1$:

$$u = \left(\frac{1}{z} - 1\right)^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}(\phi_1+\phi_2)} = \frac{X_3 + \mathrm{i}X_4}{X_1 - \mathrm{i}X_2} \qquad (81)$$

which is essentially (i.e., up to the reflection $X_2 \to -X_2$) the standard Hopf fibration of S^3. Moreover, we can calculate the energy of the soliton,

$$E = 2\pi^2 R_0^3 \mathcal{L}\left(\frac{2}{R_0^2}\right). \qquad (82)$$

Observe that the class of systems allowing for the hopfion (81) is even larger than assumed in (75). In fact, all models depending additionally on a second invariant j, with

$$j = \frac{1}{8}\mathcal{L}_4 = \frac{(\partial_\mu u \partial^\mu \bar{u})^2 - (\partial_\mu u \partial^\mu u)(\partial_\mu \bar{u} \partial^\mu \bar{u})}{(1+|u|^2)^4}, \qquad (83)$$

also possess this hopfion with unit charge. This is due to the trivial fact that the variable j can be reduced to the variable l if the eikonal equation is satisfied. As this equation already belongs to the submodel (78) one can conclude that (78) is an integrable submodel for all models of the form

$$\mathcal{L} = \mathcal{L}(l, j), \qquad (84)$$

with the non-trivial topological soliton (81). As a consequence, as

$$\mathcal{L}_{\mathrm{FN}} = 2\left(\mu^2 l - \frac{1}{e^2} j\right), \tag{85}$$

we are able to reproduce, within the generalized integrability, the exact solution for the Faddeev–Niemi model on S^3 originally found by Ward [19].

4.2 Stationary hopfions in the Faddeev–Niemi model

The topic addressed in this subsection is the existence of a time-dependent, stationary soliton in the Faddeev–Niemi model with a potential term chosen as in (49). Therefore, we consider the Lagrangian $\mathcal{L} = \frac{1}{2}\mathcal{L}_{\mathrm{FN}} - V_{\mathrm{I}}$, or, explicitly

$$\mathcal{L} = \mu^2 \frac{\partial_\mu u \partial^\mu \bar{u}}{(1+|u|^2)^2} - \frac{1}{e^2}\frac{(\partial_\mu u \partial^\mu \bar{u})^2 - (\partial_\mu u \partial^\mu u)(\partial_\mu \bar{u}\partial^\mu \bar{u})}{(1+|u|^2)^4} - \frac{\beta^2}{4}\frac{|u|^2}{(1+|u|^2)^2}, \tag{86}$$

with the field equation

$$\begin{aligned}&\mu^2(1+u\bar{u})^3\partial_\mu^2 u - 2\mu^2(1+u\bar{u})^2\bar{u}u_\mu^2 \\ &+\frac{\beta}{4}u(1-u\bar{u})(1+u\bar{u})^2 + \frac{4}{e^2}u\left[(u^\nu\bar{u}_\nu)^2 - u_\mu^2\bar{u}_\nu^2\right] \\ &-\frac{2}{e^2}(1+u\bar{u})\left[\bar{u}^{\mu\nu}u_\mu u_\nu - u^{\mu\nu}\bar{u}_\mu u_\nu + u^\mu\bar{u}_\mu\partial_\nu^2 u - u_\mu^2\partial_\nu^2\bar{u}\right] \\ &= 0 .\end{aligned} \tag{87}$$

Assuming $u = \mathrm{e}^{\mathrm{i}\omega t}v(\mathbf{r})$ results in

$$\begin{aligned}&-\mu^2(1+v\bar{v})^3\triangle v + 2\mu^2(1+v\bar{v})^2\bar{v}(\nabla v)^2 \\ &+\left(\frac{\beta}{4} - \omega^2\mu^2\right)v(1-v\bar{v})(1+v\bar{v})^2 \\ &+\frac{4}{e^2}v\left[-\omega^2(v\nabla\bar{v}+\bar{v}\nabla v)^2 + (\nabla v\cdot\nabla\bar{v})^2 - (\nabla v)^2(\nabla\bar{v})^2\right] \\ &+\frac{2\omega^2}{e^2}v(1+v\bar{v})\left[2\nabla v\cdot\nabla\bar{v} + \bar{v}\triangle v + v\triangle\bar{v}\right] \\ &-\frac{2}{e^2}(1+v\bar{v}) \\ &\quad\times\left[\bar{v}^{kj}v_k v_j - v^{kj}v_k\bar{v}_j + (\nabla v\cdot\nabla\bar{v})\triangle v - (\nabla v)^2\triangle\bar{v}\right] \\ &= 0 .\end{aligned} \tag{88}$$

Now, we assume in addition that $(\nabla v)^2 = 0$ and $\triangle v = 0$. Therefore we get

$$\begin{aligned}&\left(\frac{\beta}{4} - \omega^2\mu^2\right)v(1-v\bar{v})(1+v\bar{v})^2 \\ &+\frac{4\omega^2}{e^2}v(1-v\bar{v})(\nabla v\cdot\nabla\bar{v}) \\ &+\frac{4}{e^2}v(\nabla v\cdot\nabla\bar{v})^2 - \frac{2}{e^2}(1+v\bar{v})(\nabla v\cdot\nabla\bar{v})^j v_j \\ &= 0 ,\end{aligned} \tag{89}$$

where we used $\bar{v}^{kj}v_k v_j = (\bar{v}^k v_k)^j v_j$, which holds because of the static eikonal equation. Next, we insert the Ansatz $v = f(z)\mathrm{e}^{\mathrm{i}(m_1\phi_1 + m_2\phi_2)}$, as in the previous sections. Then the first line of (89) becomes

$$\begin{aligned}&v(1-f^2) \\ &\times\left[\left(\frac{\beta}{4} - \omega^2\mu^2\right)(1+f^2)2 + \frac{8\omega^2}{e^2R_0^2}f^2\frac{zm_1^2 + (1-z)m_2^2}{z(1-z)}\right],\end{aligned} \tag{90}$$

where we used the relation

$$\nabla v\cdot\nabla\bar{v} = \frac{2}{R_0^2}f^2\frac{zm_1^2 + (1-z)m_2^2}{z(1-z)}, \tag{91}$$

which follows from the static complex eikonal equation. In the case when $m_1 = m_2 = 1$, it leads to

$$v(1-f^2)\left[\left(\frac{\beta^2}{4} - \omega^2\mu^2\right)(1+f^2)^2 + \frac{8\omega^2}{e^2R_0^2}f^2\frac{1}{z(1-z)}\right]. \tag{92}$$

Inserting the simplest Hopf map $f = \left(\frac{1}{z} - 1\right)^{\frac{1}{2}}$, this can be rewritten as

$$\frac{v(1-f^2)}{z}\left(\frac{\beta^2}{4} - \omega^2\mu^2 + \frac{8\omega^2}{e^2R_0^2}\right). \tag{93}$$

On the other hand, the second line of (89) vanishes identically for the above profile function, as we know already from Sect. 4.1 (see also [19]).

As a result, the simplest Hopf map solves the equation of motion (87) if the following dispersion relation is satisfied:

$$\frac{\beta^2}{4} - \omega^2\mu^2 + \frac{8\omega^2}{e^2R_0^2} = 0 . \tag{94}$$

Thus, if $\mu^2 > 8/e^2R_0^2$ we obtain a stationary solution of the Faddeev–Niemi model with the new baby Skyrme potential,

$$u = \left(\frac{1}{z} - 1\right)^{1/2}\mathrm{e}^{\pm\mathrm{i}(\phi_1+\phi_2)}\mathrm{e}^{\pm\mathrm{i}\omega t}, \tag{95}$$

with the following frequency:

$$\omega^2 = \frac{\beta^2}{4}\frac{1}{\mu^2 - \frac{8}{e^2R_0^2}} . \tag{96}$$

The total energy reads

$$\begin{aligned}E &= (2\pi)^2 \\ &\times\left[\left(\mu^2R_0 + \frac{4}{e^2R_0}\right) + \frac{\beta^2}{4}\left(\frac{R_0^3}{12}\frac{\mu^2e^2R_0^2+4}{\mu^2e^2R_0^2-8} + \frac{R_0^3}{6}\right)\right].\end{aligned} \tag{97}$$

If the parameters of the model are chosen such that $\mu^2 < 8/e^2R_0^2$, then no stationary hopfion is found. However, for these parameter choices one can obtain an stationary hopfion in a slightly modified model. Namely, it is sufficient to take the potential as in (64).

There is also a special case when $\mu^2 = 8/e^2 R_0^2$. Now, in order to fulfill (94), we must set $\beta = 0$ independently of the value of ω. Therefore, now the pure Faddeev–Niemi model without any potential term is investigated. In other words, the solution (95) describes a stationary hopfion in the original Faddeev–Niemi system with arbitrary frequency. Such a Q-hopfion possesses the energy

$$E = (2\pi)^2 \frac{3}{2} \mu^2 R_0 \left[1 + \frac{\omega^2 R_0^2}{12} \right] . \tag{98}$$

As in the CP^1 model, Q-balls with unit topological charge are finite energy configurations.

4.3 Hopfions with $Q_{\mathrm{H}} = m^2$

It is possible to construct a slightly more complicated family of models, analogously to [27], which are solved by some of the other eikonal knots with higher topological charges. Concretely, we allow for the following dependence of the Lagrange density:

$$\mathcal{L}_m = \mathcal{L}(l^{(m)}) , \tag{99}$$

where

$$l^{(m)} = \sigma^{(m)}(|u|^2) \cdot \frac{\partial u \partial \bar{u}}{(1+|u|^2)^2} \tag{100}$$

and

$$\sigma^{(m)}(|u|^2) = \frac{(1+u\bar{u})^2}{u\bar{u}} \cdot \frac{(u\bar{u})^{\frac{1}{m}}}{\left(1+(u\bar{u})^{\frac{1}{m}}\right)^2} . \tag{101}$$

For $m = 1$, we just have the family of models investigated above. The equation of motion takes the form

$$\partial_\mu \left[\mathcal{L}'_m \frac{\sigma^{(m)}}{(1+|u|^2)^2} \partial^\mu \right] - \mathcal{L}'_m \frac{\partial}{\partial \bar{u}} \left[\frac{\sigma^{(m)}}{(1+|u|^2)^2} \right] = 0 , \tag{102}$$

or equivalently

$$\frac{\sigma^{(m)} \mathcal{L}''_m}{(1+|u|^2)^2} \partial_\mu l^{(m)} \partial^\mu u + \frac{\sigma^{(m)} \mathcal{L}'_m}{(1+|u|^2)^2} \partial_\mu \partial^\mu u + \mathcal{L}'_m \frac{\partial}{\partial \bar{u}} \left[\frac{\sigma^{(m)}}{(1+|u|^2)^2} \right] (\partial_\mu u)^2 = 0 , \tag{103}$$

where now the prime denoted differentiation with respect to $l^{(m)}$. Now, as before we can define a simpler submodel:

$$\partial_\mu \partial^\mu u = 0, \quad (\partial_\mu u)^2 = 0 \quad \text{and} \quad \partial_\mu l^{(m)} \partial^\mu u = 0 . \tag{104}$$

It consists of the standard pure CP^1 submodel part (the first two formulas) and an addition condition. The knotted solutions of the pure CP^1 model have been described before; see (41). Thus only the third equation in (104) needs to be solved. For this purpose we insert the static knotted solutions of the free CP^1 model, $u_k = \left(\frac{1}{z} - 1\right)^{\frac{k}{2}} \mathrm{e}^{\mathrm{i}k(\phi_1+\phi_2)}$, into (100). Then, after a simple calculation one observes that $l^{(m)}$ is constant if and only if $m = k$. In other words, each family of models with a fixed value of the parameter $m = 1, 2, 3, \ldots$, i.e., based on the variable $l^{(m)}$, possesses a topological solution:

$$u_m = \left(\frac{1}{z} - 1 \right)^{\frac{m}{2}} \mathrm{e}^{\mathrm{i}m(\phi_1+\phi_2)} , \tag{105}$$

with the Hopf index $Q_{\mathrm{H}} = m^2$.

In analogy to the $Q_{\mathrm{H}} = 1$ solution such objects can also be found in all possible models of the form

$$\mathcal{L}_m = \mathcal{L}(l^{(m)}, j^{(m)}) , \tag{106}$$

based also on the additional variable

$$j^{(m)} = (\sigma^{(m)})^2 \frac{(\partial_\mu u \partial^\mu \bar{u})^2 - (\partial_\mu u \partial^\mu u)(\partial_\mu \bar{u} \partial^\mu \bar{u})}{(1+|u|^2)^4} . \tag{107}$$

As a result, we find the interesting fact that there exists a family of modified Faddeev–Niemi models which possess exact topological knotted solitons. Namely, the models defined by the following Lagrange density:

$$\mathcal{L}^m_{\mathrm{FN}} = \mu^2 \sigma^{(m)} \frac{\partial_\mu u \partial^\mu \bar{u}}{(1+|u|^2)^2} - \frac{1}{e^2} (\sigma^{(m)})^2 \frac{(\partial_\mu u \partial^\mu \bar{u})^2 - (\partial_\mu u \partial^\mu u)(\partial_\mu \bar{u} \partial^\mu \bar{u})}{(1+|u|^2)^4} , \tag{108}$$

have the soliton solutions (105).

5 Summary and discussion

In the present paper, sigma-model type field theories with field contents parameterized by the unit three component vector field living on $S^3 \times \mathbb{R}$ space-time have been investigated. There are two reasons for choosing such a physical space-time. First of all, it stabilizes, at least for some models and for some value of the parameters, the obtained solitons by introducing a scale parameter, i.e., the radius of the sphere R_0. Moreover, it also enables us for a rather big family of models to obtain at least some solutions in exact form. Specifically, we obtain solutions in many cases where the corresponding theories on space-time $\mathbb{R}^3 \times \mathbb{R}$ either do not have solutions (like, e.g., for the $CP(1)$ model) or where there are no solutions known analytically (like, e.g., for the Faddeev–Niemi model). Whereas the first issue can be explained through Derrick's theorem, which does not hold for the three-sphere, the second one can be related to the different isometry groups of $\mathbb{R}^3$ and S^3, respectively. Indeed, the isometry group of S^3 is $SO(4)$ which has rank 2. Therefore, there exist two commuting vector fields (generators of isometries) which can be chosen to be $\mathbf{v}_1 = \partial_{\phi_1}$ and $\mathbf{v}_2 = \partial_{\phi_2}$. These are symmetry generators for all theories where the Lagrangian is a scalar; therefore the Ansatz

(12) is compatible with the equation of motion for all such theories and reduces the static equation of motion to a non-linear ODE. On the other hand, on $\mathbb{R}^3$ the isometry is only $SO(3)$ with rank 1 (forgetting the irrelevant translations). To get a second commuting vector field (e.g., the angles ξ and φ of the toroidal coordinates) one has to extend the symmetry of the model under consideration (e.g., by choosing theories with a conformally invariant static equation of motion, as for the Nicole and AFZ models; for a detailed account we refer to [34]). In more general cases like, e.g., for the Faddeev–Niemi model, only a symmetry reduction to two independent variables is possible, and the resulting non-linear PDE in two variables is still too complicated to be solved analytically.

5.1 Stability

Next we want to discuss the issue of the stability of our static solutions. For this purpose it is useful to briefly recall the situation in flat space $\mathbb{R}^3$. For actions which are homogeneous in the degree of the derivatives, a scaling instability is present and prevents the existence of soliton solutions (static solutions). This is the contents of Derrick's theorem. The instability is due to the ultraviolet (UV) collapse of field configurations when the homogeneous degree of the derivatives is less than three, and due to infrared (IR) collapse for a homogeneous degree greater than three. For a homogeneous degree exactly equal to three the energy of a static field configuration is invariant under scaling, and static solutions may exist. In addition, the group of base space symmetries of the static equation of motion is enhanced (e.g., conformal symmetries instead of isometries). This is exactly what happens, e.g., in the Nicole and AFZ models. If, on the other hand, the theory consists of a sum of terms with different degrees of derivatives such that at least one has degree less than three, and at least one has degree greater than three, then these two terms scale oppositely under scale transformations, and static solutions may exist. Further, the model is not scale invariant; therefore solitons have a typical "size" as an intrinsic property. This is the case, e.g., for the Faddeev–Niemi model.

Now let us discuss the analogous situation on the sphere S^3. On the three-sphere an IR collapse is no longer possible. A field configuration which obeys a non-trivial boundary condition (i.e., which has a non-zero topological index) will always contribute some non-zero values of derivatives over some finite subvolumes of the entire S^3. On the other hand, an UV collapse (shrinking of field configurations) is still possible. Therefore, we expect that theories with actions which are homogeneous with less than three derivatives will not have genuine solitons – i.e., static solutions which are absolute minima of the energy within a sector with fixed topological charge. On the other hand, for models which contain at least one term with degree in derivatives greater than three we expect stable solutions, i.e., genuine solitons.

We want to investigate the issue of stability more closely for the simplest Hopf map (81), which solves most of the theories we have studied in this paper. The stability issue of this field configuration has already been investigated in [19] for the Faddeev–Niemi model, so we can make use of these results. In [19] a one-parameter family of fields u_λ has been constructed, where $\lambda = 1$ just gives the standard Hopf map. The energy density of the standard Hopf map is constant both for the quadratic lagrangian $\mathcal{L}_2$ (the CP^1 term) and for the quartic lagrangian $\mathcal{L}_4$. Further, both energy densities become peaked for very large or very small values of the parameter λ (around the north pole or south pole of the S^3, respectively). The energy of the quadratic (CP^1) term for the one-parameter family u_λ has a maximum at $\lambda = 1$. For very small or very large values of λ the energy approaches zero. However, the limiting field configurations for $\lambda = 0$ or $\lambda = \infty$ cannot be attained, because they are trivial and do not belong to the sector with Hopf index one. Therefore, there does not exist a genuine soliton in the sector with Hopf index one for the CP^1 model. The solution for $\lambda = 1$ is a saddle point solution rather than a minimum. For the quartic term, the energy has a minimum at $\lambda = 1$. Further, the energy tends to infinity in the limits $\lambda \to 0$ and $\lambda \to \infty$. This supports the conjecture that the standard Hopf map is a genuine soliton (minimizer of the sector with Hopf index one) for the quartic model (although there does not seem to exist a rigorous proof up to now). For the case of the Faddeev–Niemi model $\mathcal{L}_{\rm FN} = \mathcal{L}_2 - \mathcal{L}_4$ (here we ignore constants) we just briefly repeat the discussion of [19]. For sufficiently small radius R_0 of the three-sphere the energy of the quartic term dominates (behaving like $1/R_0$), and the energy is minimized for $\lambda = 1$. So probably the standard Hopf map is a true minimum. For large values of the sphere radius the energy of the quadratic term dominates (behaving like R_0), and the standard Hopf map is just a saddle point. However, now, complete UV collapse is not possible (this would render the energy of the quartic term infinite). Instead the energy is minimized for some finite $\lambda_0 \geq 1$ (or, equivalently, for its inverse $1/\lambda_0$) with the energy density localized around the north pole (or south pole) of the S^3. For larger values of R_0 the localization becomes more pronounced (i.e., λ_0 becomes larger). So a true soliton probably exists for the Faddeev–Niemi model even for large values of the sphere radius, but it is no longer the standard Hopf map with its energy density evenly distributed over the whole S^3. We expect this generic pattern of stability also to hold for higher Hopf index. The generalization of the above discussion of stability to the other models studied in this paper is straightforward.

Finally, let us just mention that the question of stability is more involved for the stationary solutions (Q-balls). Firstly, stability is no longer related to the minimization of the energy and, secondly, the presence of further non-trivial conserved charges (like the Noether charge in Sect. 3.2) complicates the analysis and tends to make solutions more stable. A detailed discussion of that issue is beyond the scope of this article.

5.2 Summary of results

Firstly, static knotted configurations solving the complex eikonal equation have been derived. They are the S^3 counterparts of the eikonal knots on $\mathbb{R}^3$ and, therefore, describe linked torus knots with arbitrary value of the topological charge. The problem whether non-torus knots, represented for instance by the figure-eight knot, can also be found for the eikonal equation is still an open question. Unfortunately, our method does not allow us construct such knots. In addition, time-dependent knots (stationary or exploding/collapsing ones) have been constructed.

Secondly, we have shown that eikonal knots with Hopf index $Q_{\mathrm{H}} = \pm m^2$, where $m \in Z$, are solutions of the pure CP^1 model on S^3. The energies of these solutions can be related to their topological charges. Concretely, the energy is proportional to the square root of the charge. Stability analysis shows that these solutions are not stable, i.e., they are not true solitons. Instead, they are saddle point solutions. A family of exact stationary solutions has been obtained, as well, for the CP^1 model with the "new baby Skyrme" potential term. Their frequencies are strictly determined and do not depend on the topology of the solutions (value of the parameters m_1 and m_2). Moreover, a slight modification of the potential gives collapsing/exploding solutions.

Thirdly, in a very large class of models a static hopfion with unit Hopf index (the standard Hopf fibration) has been found. In the case of the Faddeev–Niemi model, we reproduced a solution already obtained by Ward [19]. In addition, a stationary generalization of the soliton has been derived for the Faddeev–Niemi model with the "new baby Skyrme" potential. Its frequency is determined by the parameters of the model. This may be of some interest in the context of the effective model for the low-energy quantum gluodynamics. Namely, the Faddeev–Niemi model spontaneously breaks the global $O(3)$ symmetry and, as a consequence, two massless Goldstone bosons appear. To get rid of such non-physical excitations one has to improve the model and add a symmetry-breaking term [31] (see also [32]). The most obvious way to accomplish this is to introduce a potential. For the special case when the parameters obey $\mu^2 = 8/e^2 R_0^2$ a stationary hopfion with $Q_{\mathrm{H}} = 1$ has been found in the pure Faddeev–Niemi model, where the frequency may take on arbitrary values.

Finally we have proved that also more complicated static hopfions with higher values of the topological charge can be obtained in modified models. The modification is given by the so-called dielectric function.

There are several directions in which our work can be continued. One could, for example, try to derive static soliton solutions in the models with the new baby Skyrme potential added and compare them with the stationary solutions.

On the other hand, one could study the issue of quantization of the obtained Hopf solitons [35]. Also the relevance of the saddle point solutions of the CP^1 model for its subsequent quantization would be worth investigating.

Finally, we hope that the results presented here lead to some further insight into general properties of theories with knotted solitons and may, in this respect, also help in understanding the corresponding theories in standard Minkowski space-time.

Acknowledgements. This research was partly supported by MCyT(Spain) and FEDER (FPA2005-01963), Incentivos from Xunta de Galicia and the EC network "EUCLID". Further, C.A. acknowledges support from the Austrian START award project FWF-Y-137-TEC and from the FWF project P161 05 NO 5 of N.J. Mauser. A.W. gratefully acknowledges support from the Polish Ministry of Education and Science.

References

1. E. Babaev, L.D. Faddeev, A. Niemi, Phys. Rev. B **65**, 100 512 (2002); E. Babaev, Phys. Rev. Lett. **89**, 067 001 (2002)
2. Y.M. Cho, H. Khim, P. Zhang, Phys. Rev. A **72**, 063 603 (2005)
3. G.'t Hooft, Nucl. Phys. B **79**, 276 (1974); G. 't Hooft, Nucl. Phys. B **153**, 141 (1979); A. Polyakov, Nucl. Phys. B **120**, 429 (1977)
4. L. Faddeev, A. Niemi, Nature **387**, 58 (1997)
5. Y.M. Cho, Phys. Rev. Lett. **87**, 252 001 (2001)
6. L. Faddeev, A. Niemi, Phys. Rev. Lett. **82**, 1624 (1999); E. Langmann, A. Niemi, Phys. Lett. B **463**, 252 (1999)
7. Y.M. Cho, Phys. Rev. D **21**, 1080 (1980); Y.M. Cho, Phys. Rev. D **23**, 2415 (1981); Y.M. Cho, Phys. Rev. Lett. **46**, 302 (1981)
8. S.V. Shabanov, Phys. Lett. B **463**, 263 (1999); S.V. Shabanov, Phys. Lett. B **458**, 322 (1999)
9. K.-I. Kondo, Phys. Lett. B **600**, 287 (2004); K.-I. Kondo, T. Murakami, T. Shinohara, Prog. Theor. Phys. **115**, 201 (2006); K.-I. Kondo, T. Murakami, T. Shinohara, Eur. Phys. J. C **42**, 475 (2005)
10. Y.M. Cho, Phys. Lett. B **616**, 101 (2005)
11. L. Dittmann, T. Heinzl, A. Wipf, JHEP **0212**, 014 (2002)
12. H. Gies, Phys. Rev. D **63**, 125 023 (2001)
13. F. Lin, Y. Yang, Comm. Math. Phys. **249**, 273 (2004)
14. R.A. Battye, P.M. Sutcliffe, Phys. Rev. Lett. **81**, 4798 (1998); R.A. Battye, P.M. Sutcliffe, Proc. Roy. Soc. Lond. A **455**, 4305 (1999)
15. J. Hietarinta, P. Salo, Phys. Lett. B **451**, 60 (1999); J. Hietarinta, P. Salo, Phys. Rev. D **62**, 81 701 (2000); J. Hietarinta, J. Jäykkä, P. Salo, Phys. Lett. A **321**, 324 (2004)
16. D.A. Nicole, J. Phys. G **4**, 1363 (1978)
17. H. Aratyn, L.A. Ferreira, A.H. Zimerman, Phys. Lett. B **456**, 162 (1999); H. Aratyn, L.A. Ferreira, A.H. Zimerman, Phys. Rev. Lett. **83**, 1723 (1999)
18. S. Deser, M.J. Duff, C.J. Isham, Nucl. Phys. B **114**, 29 (1976)
19. R.S. Ward, Nonlinearity **12**, 241 (1999)
20. E. De Carli, L.A. Ferreira, J. Math. Phys. **46**, 012 703 (2005)
21. O. Alvarez, L.A. Ferreira, J. Sánchez-Guillén, Nucl. Phys. B **529**, 689 (1998)
22. L.A. Ferreira, J. Sánchez-Guillén, Phys. Lett. B **504**, 195 (2001); J. Sánchez-Guillén, Phys. Lett. B **548**, 252 (2002); C. Adam, J. Sánchez-Guillén, JHEP **0501**, 004 (2005)
23. A. Wereszczyński, Phys. Lett. B **621**, 201 (2005); C. Adam, J. Sánchez-Guillén, Phys. Lett. B **626**, 235 (2005)
24. C. Adam, J. Sánchez-Guillén, A. Wereszczyński, hep-th/0511277
25. C. Adam, J. Math. Phys. **45**, 4017 (2004)
26. A. Wereszczyński, Eur. Phys. J. C **42**, 461 (2005)

27. A. Wereszczyński, Eur. Phys. J. C **41**, 265 (2005)
28. R.S. Ward, Phys. Lett. B **158**, 424 (1985)
29. A.E. Kudryavtsev, B. M.A.G. Piette, W.J. Zakrzewski, Nonlinearity **11**, 783 (1998); T. Weidig, hep-th/9811238; T. Weidig, in: Erice 1999, Basics and Highlights in Fundamental Physics, p. 620–629; T.A. Ioannidou, V.B. Kopeliovich, W.J. Zakrzewski, J. Exp. Theor. Phys. **95**, 572 (2002)
30. A. Wereszczyński, Mod. Phys. Lett. A **20**, 1135 (2005)
31. L. Faddeev, A. Niemi, Phys. Lett. B **525**, 195 (2002)
32. L. Dittmann, T. Heinzl, A. Wipf, Nucl. Phys. B (Proc. Suppl.) **106**, 649 (2002); Nucl. Phys. B (Proc. Suppl.) **108**, 63 (2002)
33. R. Leese, Nucl. Phys. B **366**, 283 (1991)
34. O. Babelon, L.A. Ferreira, JHEP **0211**, 020 (2002)
35. W.C.M. Su, Chin. J. Phys. **40**, 516 (2002); W.C.M. Su, Phys. Lett. B **525**, 201 (2002); S. Krusch, J.M. Speight, hep-th/0503067